A Handbook for

Engineering Mathematics

Contains key theory concepts, formulae
and practice problems for

GATE

Also useful for ESE & other competitive examinations

by

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MADE EASY Publications

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A Handbook for Engineering Mathematics

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Director's Message

B. Singh (Ex. IES)

When the topic of completion of subjects comes while preparing for competitive exams, then studying one extra subject called as MATHEMATICS is often a tough pill to swallow. This is mainly due to the time constraints; as in this competitive environment when everybody is toiling, there is a lot to do in a limited time frame.

As it is rightly said," Mathematics is not about numbers, equations, computations or algorithms it is about understanding." Understanding mathematics is not as easy as it is said; to simplify this easy to say but difficult to be done task, the MADE EASY team has come up with this Handbook of Mathematics which contains all formulae and theoretical concepts of Engineering Mathematics.

And as we all know" the only way to learn mathematics is to do mathematics'', so to facilitate all aspirants we have incorporated practice problems for GATE, which will help you to strengthen the concepts and gain confidence. This book will act as a two in one tool for preparation, initially will help in preparing the subject and later will serve as a revision aid with all formulae at one place.

I acknowledge the sincere efforts of Mr. D.V. Sridhar and hope this book will assist in preparation of GATE, ESE and other competitive examinations.

B. Singh (Ex. IES)

CMD, MADE EASY Group

A Handbook for

Engineering Mathematics

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Engineering Mathematics

1

Basic Concepts

CONTENTS

Basic Concepts

1

I Elementary Algebra

1. Powers and Roots

(i) $a^0 = 1;\ a \neq 0$

(ii) $a^m a^n = a^{m+n}$

(iii) $\dfrac{a^m}{a^n} = a^{m-n}$

(iv) $(ab)^m = a^m b^m$

(v) $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$

(vi) $\left(a^m\right)^n = a^{mn}$

(vii) $a^{-m} = \dfrac{1}{a^m}$

(viii) $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$

(ix) $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$

(x) $a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

2. Logarithms

Definition: $y = \log_a(x)$ if and only if $a^y = x$ where $a, x > 0$ and $a \neq 1$.

Natural logarithm: $e^y = x$ if and only if $y = \log_e(x) = \ln(x)$

Where $e = \lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x = 2.71828182846\ldots$

(i) $\log_a 1 = 0$

(ii) $\log_a a = 1$

(iii) $\log_a(mn) = \log_a m + \log_a n$

(iv) $\log_a\left(\dfrac{m}{n}\right) = \log_a m - \log_a n$

(v) $\log_a\left(m^n\right) = n\log_a m$

(vi) $\log_b a = \dfrac{1}{\log_a b}$

(vii) $\log_{(a^k)}(m) = \dfrac{1}{k}\log_a m$

(viii) $\log_a m = \log_b m \cdot \log_a b$ where $b > 0$ and $b \neq 1$

(ix) $\log_a m = \dfrac{\log_b m}{\log_b a}$

(x) $x^{\log_a y} = y^{\log_a x}$

(xi) $x = x^{\log_a a} = a^{\log_a x}$

(xii) $x = e^{\ln x} = \ln e^x$

3. Binomial Theorem

(i) Factorials

(a) $n! = 1\times 2\times 3\times \ldots \times (n-1)\times n$ (b) $0! = 1! = 1$

(ii) Binomial Coefficient ${}^nC_r = \dfrac{n!}{r!(n-r)!}$

(iii) Binomial Theorem

$$(x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + \ldots + {}^nC_ny^n$$

(iv) Product Formulas

(a) $(a+b)^2 = a^2 + 2ab + b^2$

(b) $(a-b)^2 = a^2 - 2ab + b^2$

(c) $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

(d) $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

(v) Factoring Formulas

(a) $a^2 - b^2 = (a-b)(a+b)$

(b) $a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$

(c) $a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right)$

(d) $a^{2n} - b^{2n} = \left(a^n - b^n\right)\left(a^n + b^n\right)$

(e) $a^n - b^n = (a-b)\left(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + ab^{n-2} + b^{n-1}\right)$

Example: $\left(1-x^n\right) = (1-x)\left(1 + x + x^2 + x^3 + \ldots + x^{n-1}\right)$

(f) If n is odd then,

$$a^n + b^n = (a+b)\left(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \ldots - ab^{n-2} + b^{n-1}\right)$$

Example:

(g) $a^5 - b^5 = (a-b)\left(a^4 + a^3b + a^2b^2 + ab^3 + b^4\right)$

(h) $a^5 + b^5 = (a+b)\left(a^4 - a^3b + a^2b^2 - ab^3 + b^4\right)$

4. Sequences

(i) Arithmetic sequence

$a, a+d,$ a+$2d, a+3d, \ldots$

n^{th} term $= t_n = a+(n-1)d$

Sum to n terms $= S_n = \frac{n}{2}[2a+(n-1)d]$

(ii) Geometric sequence:

$a, ar, ar^2, ar^3, \ldots$

n^{th} term $= t_n = ar^{n-1}$

$$\text{Sum to } n \text{ terms} = S_n = \begin{cases} \dfrac{a(r^n-1)}{(r-1)} & r>1 \\ \dfrac{a(1-r^n)}{(1-r)} & r<1 \\ na & r=1 \end{cases}$$

(iii) Sum to infinite terms of geometric sequence

$$S_\infty = a+ar+ar^2+\ldots = \frac{a}{1-r} \qquad -1<r<1$$

5. Mean Values of n real numbers

$$a_1, a_2, \ldots, a_n$$

(i) Arithmetic mean: $\dfrac{a_1+a_2+\ldots+a_n}{n}$

(ii) Geometric mean: $(a_1.a_2....a_n)^{1/n}$

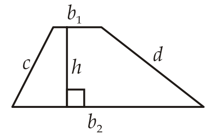
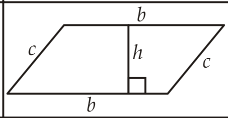
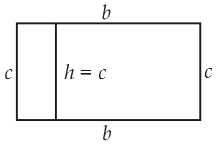
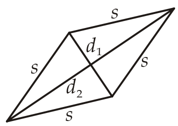
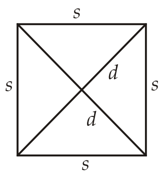
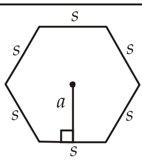
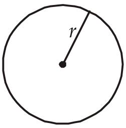
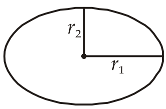
(iii) Harmonic mean: $\dfrac{n}{(1/a_1)+(1/a_2)+\ldots+(1/a_n)}$

6. Formulas for summation

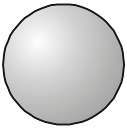
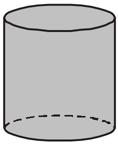
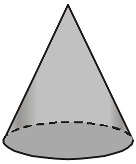
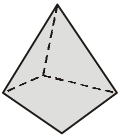
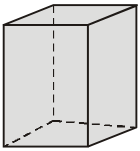
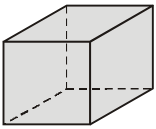
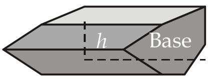
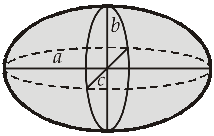
1. $1+2+\ldots+n = \dfrac{n(n+1)}{2}$
2. $1+3+\ldots+(2n-1) = n^2$
3. $2+4+\ldots+(2n) = n(n+1)$
4. $1^2+2^2+\ldots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$
5. $1^3+2^3+\ldots+n^3 = \dfrac{n^2(n+1)^2}{4}$

II Geometry

1. Two-Dimensional Geometry

Shape	Figure	Perimeter	Area
Trapezoid		$P = b_1 + b_2 + c + d$ b_1, b_2 = bases c, d = sides	$A = \frac{1}{2}(b_1 + b_2)\,h$ b_1, b_2 = bases h = height
Parallelogram		$P = 2b + 2c$ b, c = sides	$A = bh$ b = base h = height
Rectangle		$P = 2b + 2c$ b, c = sides	$A = bh$ b = base h = height
Rhombus		$P = 4s$ s = side	$A = bh = \frac{1}{2}(d_1 d_2)$ d_1, d_2 = diagonals
Square		$P = 4s$ s = side	$A = s^2 = \frac{1}{2}d^2$ d = diagonals
Regular Polygon		$P = ns$ n = number of sides s = side	$A = \frac{1}{2}a \cdot P$ a = apothem P = perimeter
Circle		$C = 2\pi r = \pi d$ r = radius d = diameter	$A = \pi r^2$ r = radius
Ellipse		$P \approx 2\pi\sqrt{\frac{1}{2}(r_1^2 + r_2^2)}$ r_1 = major axis radius r_2 = minor axis radius	$A = \pi r_1 r_2$ r_1 = major axis radius r_2 = minor axis radius

2. Three-Dimensional Geometry

Shape	Figure	Surface Area	Volume
Sphere		$SA = 4\pi r^2$ r = radius	$V = \frac{4}{3}\pi r^3$ r = radius
Right Cylinder		$SA = 2\pi rh + 2\pi r^2$ h = height r = radius of base	$V = \pi r^2 h$ h = height r = radius of base
Cone		$SA = \pi rl + \pi r^2$ l = slant height r = radius of base	$V = \frac{1}{3}\pi r^2 h$ h = height r = radius of base
Square Pyramid		$SA = 2sl + s^2$ s = base side length l = slant height	$V = \frac{1}{3}s^2 h$ *s = base side length* *h = height*
Rectangular Prism		$SA = 2\,(lw + lh + wh)$ l = length w = width h = height	$V = lwh$ l = length w = width h = height
Cube		$SA = 6s^2$ s = side length (all sides)	$V = s^3$ s = side length (all sides)
General Right Prism		$SA = Ph + 2B$ P = Perimeter of base h = height (or length) B = area of base	$V = Bh$ B = area of base h = height
Ellipsoid		–	$\frac{4}{3}\pi\,abc$

III Analytic Geometry

1. 2D-Coordinate system

(i) Distance between Two points (x_1, y_1) and (x_2, y_2)

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

(ii) The point of division of the line joining (x_1, y_1) and (x_2, y_2) in the ratio $m : n$ is

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$$

(iii) Midpoint of the line joining (x_1, y_1) and (x_2, y_2) is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

(iv) Area of triangle formed the vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3), is

$$A = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Note:

☑ The sign is chosen so that the area is nonnegative.

☑ If the area is zero, then three points A, B and C are collinear (lie on same line)

(v) Distance between two points (r_1, θ_1) and (r_2, θ_2) is in polar coordinates is

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)}$$

(vi) Equations of transformation

(a) Cartesian to polar coordinates is

$x = r\cos\theta$ and $y = r\sin\theta$

(b) Polar coordinates to Cartesian coordinates

$$r = \sqrt{x^2 + y^2} \text{ and } \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

2. Straight line

(i) Slope of line

(a) Slope (m) of line passing through the points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} \qquad x_1 \neq x_2$$

Inclination *of the line*: The angle made by the line with the positive direction of x-axis

(b) Slope (gradient) (m) of line with inclination θ is

$$m = \tan\theta,\ \ \theta \neq 90°$$

(c) Slope of line $ax + by + c = 0$ is $-\frac{a}{b}$

(ii) Equation of line

(a) Point-slope form: $(y - y_1) = m\,(x - x_1)$ where slope $= m$ and (x_1, y_1) is the point

(b) Point-point form: $(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ where the points are (x_1, y_1) and (x_2, y_2)

(c) Slope-Intercept Form: $y = mx + c$ where slope $= m$ and y-intercept $= c$

(d) Intercept form: $\frac{x}{a} + \frac{y}{b} = 1$ where x-intercept $= a$ and y-intercept $= b$

(e) Point-inclination form: $\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$ where inclination $= \theta$ and (x_1, y_1) is the point

(f) Normal form: $x\cos\theta + y\sin\theta = p$

- p = Length of the perpendicular (normal) from origin to the line.
- θ = Angle of Inclination of normal with the positive direction of x-axis.

(g) Vertical line $x = a$

(h) Horizontal line $y = b$

(iii) Results

(a) Angle between two lines having slopes m_1 and m_2 is $\tan^{-1}\frac{m_1 - m_2}{1 + m_1 m_2}$.

(b) Two lines are parallel if $m_1 = m_2$.

(c) Two lines are perpendicular if $m_1 \cdot m_2 = -1$.

(d) General Equation of Line is $ax + by + c = 0$.

(e) Equation of Line parallel to is $ax + by + c = 0$ is $ax + by + k = 0$.

(f) The length of perpendicular from (x_1, y_1) to the line $ax + by + c = 0$

is $\left|\dfrac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$.

3. Circle

A circle is the set of all points in a plane that are equidistant from a fixed point in the plane.

(i) The equation of circle having center (h, k) and radius r is

$$(x - h)^2 + (y - k)^2 = r^2$$

(ii) The equation of circle with center at origin and radius r is $x^2 + y^2 = r^2$

(iii) Parametric equations of circle $x^2 + y^2 = r^2$ are

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

(iv) The general Equation of circle: $x^2 + y^2 + 2gx + 2fy + c = 0$;

Center $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$

4. Conic Sections

(i) General equation of conic

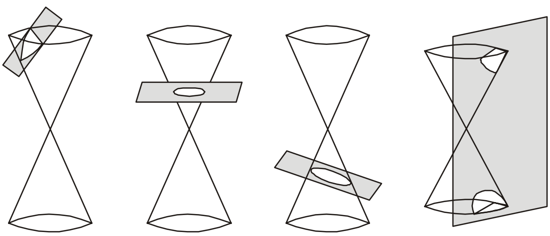
The Equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$ represents

Ellipse if $b^2 - 4ac < 0$

Parabola if $b^2 - 4ac = 0$

Hyperbola if $b^2 - 4ac > 0$

(ii) Conic sections are the curves obtained by intersecting a right circular cone by a plane.



Parabola Circle Ellipse Hyperbola

(iii) A conic section is the locus of a point P which moves so that its distance from a fixed point [**Focus** S] is always in a constant ratio [**eccentricity** e] to its perpendicular distance from a fixed line [**Directrix**].

The conic is called

- **Ellipse** if $e < 1$
- **Parabola** if $e = 1$
- **Hyperbola** if $e > 1$

(a) **Principal axis:** A straight line passing through the focus and perpendicular to the directrix.

Note:

☑ Conic is symmetrical about Principal axis.

(b) **Vertex:** The points of intersection of a conic and its principal axis.

Note:

☑ A conic has at most two vertices.

(c) **Centre:** The point which bisects every chord of a conic passing through it.

Note:

☑ If a conic has only one vertex then its centre coincides with the vertex.

(d) **Focal chord:** A chord passing through the focus.

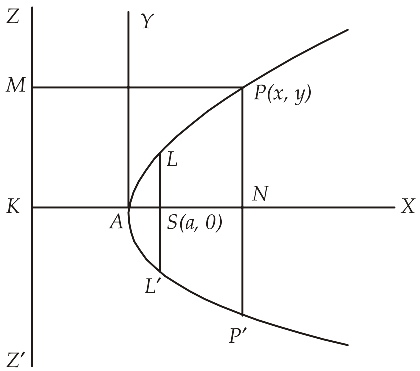
(e) **Latus rectum:** The focal chord which is perpendicular to principal axis.

(f) **Double ordinate:** A chord of the conic which is perpendicular to principal axis.

4. Parabola

The locus of a point (P) whose distance from a fixed point (S) bears a constant ratio (e =1) to its distance from a fixed line (KZ) is called a parabola.

(i) The standard formula of a parabola: $y^2 = 4ax$



Focus: $S(a, 0)$

Center: $A(0, 0)$

Principal axis: $y = 0$ (x-axis)

Eccentricity: $e = 1$

Length of Latus rectum: $LL' = 4a$

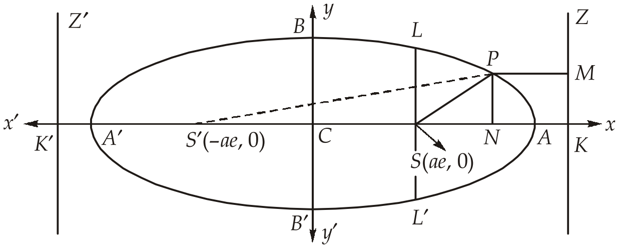
Equation of the directrix: $x + a = 0$

(ii) Parametric equations of the parabola: $x = at^2 \quad ; \quad y = 2at$

5. Ellipse

The locus of a point (P) whose distance from a fixed point (S and S') bears a constant ratio ($e < 1$) to its distance from a fixed line ((KZ and $K'Z'$) is called an ellipse.

(i) Standard Equation of an ellipse: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$



Major axis: The line joining the two vertices A' and A

Minor axis: The line passing through the centre perpendicular to the major axis, i.e., BB'

Principal axis: Major axis and Minor axis

Length of Major axis: $2a$

Length of Minor axis: $2b$

Eccentricity: $e = \dfrac{\sqrt{a^2 - b^2}}{a}$

Foci: $S'(-ae, 0)$ and $A'(a, 0)$

Vertices: $A(-a, 0)$ and $A'(a, 0)$

Length of Latus rectum: $LL' = \dfrac{2b^2}{a}$

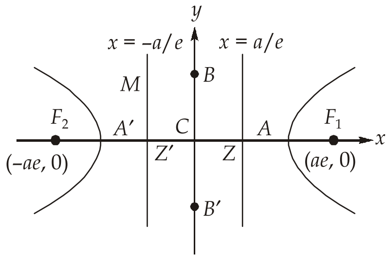
Equation of the directrix: $x = \pm\dfrac{a}{e}$

(ii) Parametric equations of the ellipse: $x = a\cos t \quad ; \quad y = b\sin t$

6. Hyperbola

The locus of a point (P) whose distance from a fixed point (F_1 and F_2) bears a constant ratio ($e > 1$) to its distance from a fixed line is called a hyperbola.

(i) The standard formula of a hyperbola: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$



Transverse axis: The line segment AA' joining the vertices

Conjugate axis: The line segment joining the points $B(0, b)$ and B' $(0, -b)$ **Principal axis:** Major axis and Minor axis

Length of Transverse axis: $2a$

Length of Conjugate axis: $2b$

Equation of Transverse axis: $y = 0$ (x-axis)

Equation of Conjugate axis: $x = 0$ (y-axis)

Vertices: $A(a, 0)$ and $A'(-a, 0)$

Eccentricity: $e = \dfrac{\sqrt{a^2 + b^2}}{a}$

Foci: $F_2(-ae, 0)$ and $F_1(ae, 0)$

Equation of the directrix: $x = \pm\dfrac{a}{e}$

Length of Latus rectum LL': $LL' = \dfrac{2b^2}{a}$

(ii) Parametric equations of the hyperbola: $x = a\sec t \;;\; y = b\tan t$

7. Planes in three dimensions

(i) General form:

$$Ax + By + Cz + D = 0$$

where direction (A, B, C) is normal to the plane.

(ii) Intercept form:

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

this plane passes through the points $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$.

(iii) Three point form

$$\begin{vmatrix} x - x_3 & y - y_3 & z - z_3 \\ x_1 - x_3 & y_1 - y_3 & z_1 - z_3 \\ x_2 - x_3 & y_2 - y_3 & z_2 - z_3 \end{vmatrix} = 0$$

(iv) Angle between two planes:

The angle between two planes:

$$A_1x + B_1y + C_1z + D_1 = 0$$
$$A_2x + B_2y + C_2z + D_2 = 0$$

$$\cos\theta = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + A_2^2 + A_3^2}\sqrt{B_1^2 + B_2^2 + B_3^2}}$$

Note:

☑ The Planes are parallel if and only if $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$

☑ The Planes are perpendicular if and only if

$$A_1A_2 + B_1B_2 + C_1C_2 = 0$$

(v) The Distance of $P(x_1, y_1, z_2)$ from the plane $Ax + By + Cz + D = 0$ is

$$d = \frac{|Ax_1 + Bx_2 + Cx_3 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

IV Elementary Functions

A function f from set A to set B ($f: A \to B$) is a rule which assigns every element of A to a unique element of B.

Where A is domain; B is co-domain; $f(A)$ is range

(i) Monotonicity (A monotonic function preserves or reverses the given order) $f(x)$ is

(a) Monotonically increasing if $m \le n \Rightarrow f(m) \le f(n)$.

(b) Monotonically decreasing if $m \le n \Rightarrow f(m) \ge f(n)$.

(c) Strictly increasing if $m < n \Rightarrow f(m) < f(n)$.

(d) Strictly decreasing if $m < n \Rightarrow f(m) > f(n)$.

(ii) Bounded Function: $m \le f(x) \le M$ for all $x \in$ Domain

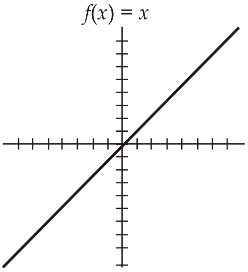
(iii) Even Function: $f(-x) = f(x)$ Graph is symmetric about y axis

(iv) Odd Function: $f(-x) = -f(x)$ Graph is symmetric about origin

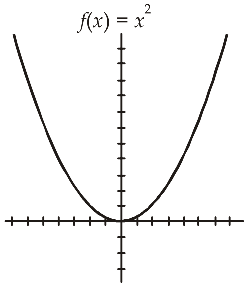
(v) Periodic Function of period T: $f(x + T) = f(x)$

(vi) Basic Functions

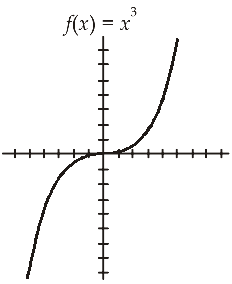
(a) Identity Function



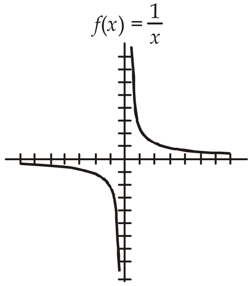
(b) Squaring Function



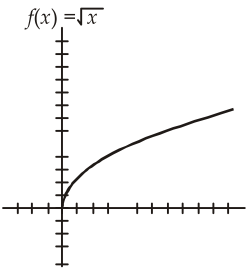
(c) Cubing Function



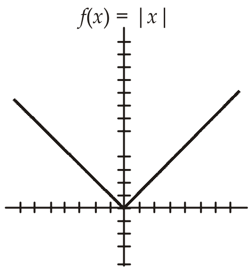
(d) Reciprocal Function



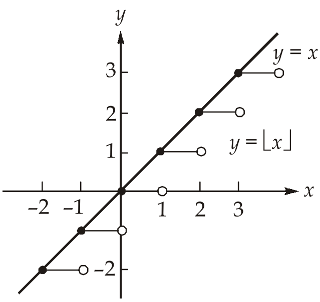
(e) Square Root Function



(f) Absolute Value Function



(g) Greatest Integer function (int(x) or [x]): Greatest integer less than or equal to x

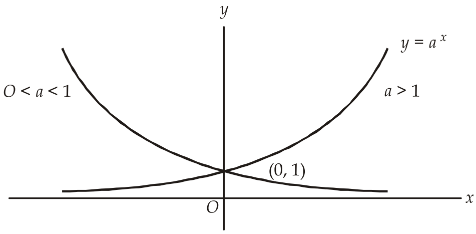


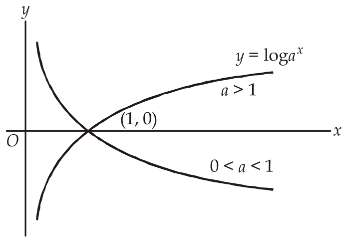
(h) Exponential Function $y = a^x \quad a > 0 \quad (a \neq 1)$

(i) Logarithmic Function

$y = \log_a x \quad a > 0 \quad (a \neq 1)$

$y = \log_e x = \ln x$





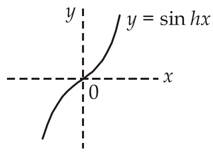
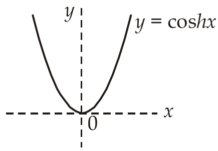
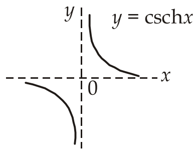
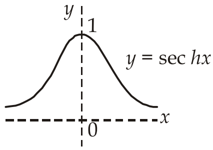
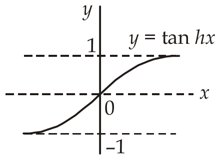
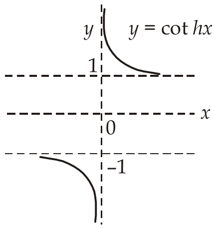
(vii) Properties

Function	Domain	Range	Symmetry	Bounded	Increasing	Decreasing
1. Identity function $y = x$	$(-\infty, +\infty)$ or all x	$(-\infty, +\infty)$ or all y	Odd	No	$(-\infty, +\infty)$ or all x	None
2. Squaring Function $y = x^2$	$(-\infty, +\infty)$ or all x	$[0, \infty)$ or $y \geq 0$	Even	Bounded Below	$(0, +\infty)$ or $x > 0$	$(-\infty, 0)$ or $x < 0$
3. Cubing Function $y = x^3$	$(-\infty, +\infty)$ or all x	$(-\infty, +\infty)$ or all y	Odd	No	$(-\infty, +\infty)$	None
4. Reciprocal Function $y = \frac{1}{x}$	$(-\infty, 0) \cup (0, \infty)$ or $x \neq 0$	$(-\infty, 0) \cup (0, \infty)$ or $y \neq 0$	Odd	No	None	$(-\infty, 0) \cup (0, \infty)$ or $x \neq 0$
5. Square root Function $y = \sqrt{x}$	$[0, \infty)$ or $x \geq 0$	$[0, \infty)$ or $y \geq 0$	None	Bounded Below	$[0, \infty)$ or $x \geq 0$	None
6. Exponential Function $y = e^x$	$(-\infty, +\infty)$ or all x	$(0, +\infty)$	None	Bounded Below	$(-\infty, +\infty)$ or all x	None
7. Logarithm Function $y = \ln x$	$(0, +\infty)$ or $x > 0$	$(-\infty, +\infty)$ or all y	None	None	$(0, +\infty)$ or $x > 0$	None
8. Absolute Value Function $y = \lvert x \rvert$	$(-\infty, +\infty)$ or all x	$[0, \infty)$ or $y \geq 0$	Even	Bounded Below	$(0, \infty)$ or $x > 0$	$(-\infty, 0)$ or $x < 0$
9. Greatest integer Function $y = \text{int}(x = [x])$	$(-\infty, +\infty)$ or all x	Z (Set of integers)	None	None	None	None

(viii) Polynomial Function

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

(ix) Hyperbolic functions

Hyperbolic function	Domain	Range	Graph
1. $\sinh x = \dfrac{e^x - e^{-x}}{2}$	$(-\infty, +\infty)$ or all x	$(-\infty, +\infty)$ or all y	
2. $\cos h\, x = \dfrac{e^x + e^{-x}}{2}$	$(-\infty, +\infty)$ or all x	$[1, \infty)$ or $y \geq 1$	
3. $\csc hx = \dfrac{2}{e^x - e^{-x}}$	$(-\infty, 0) \cup (0, \infty)$ or $x \neq 0$	$(-\infty, 0) \cup (0, \infty)$ or $y \neq 0$	
4. $\sec hx = \dfrac{2}{e^x + e^{-x}}$	$(-\infty, +\infty)$ or all x	$(0, 1]$ or $0 < y \leq 1$	
5. $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$	$(-\infty, +\infty)$ or all x	$(-1, +1)$ or $\lvert y \rvert < 1$	
6. $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$	$(-\infty, 0) \cup (0, \infty)$ or $x \neq 0$	$(-\infty, -1) \cup (1, \infty)$ $\lvert y \rvert > 1$	

(x) Function Transformations

Vertical Translation

$y = f(x) + c$ Shift the graph up c units.

$y = f(x) - c$ Shift the graph down c units.

Horizontal Translation

$y = f(x + c)$ Shift the graph left c units.

$y = f(x - c)$ Shift the graph right c units.

Reflection

$y = -f(x)$ Reflect across the x-axis.

$y = f(-x)$ Reflect across the y-axis.

Vertical Stretch or Shrink

$y = c \cdot f(x)$ Stretch by a factor of c if $c > 1$.

Shrink by a factor of c if $c < 1$.

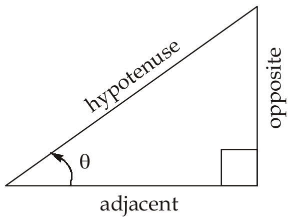
V Trigonometry

1. Definition of Trigonometric Functions

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

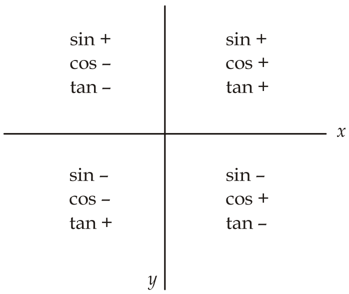


2. Trigonometric Functions of common angles

θ	0	30	45	60	90
sin θ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan θ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

3. Signs of Trig Functions by Quadrant (Quadrant rule)

Signs of Trig Function by Quadrant



$$\text{fn}(n.\pi/2+\theta) = \begin{cases} \pm fn(\theta) & n \text{ is even} \\ \pm co-fn(\theta) & n \text{ is odd} \end{cases}$$

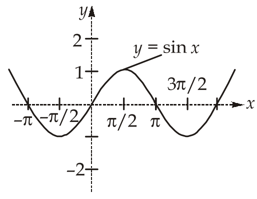
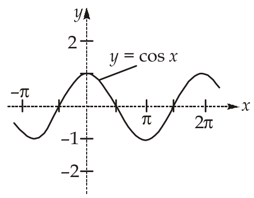
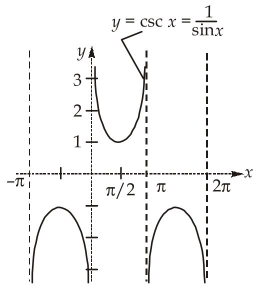
The ± sign is decided by quadrant rule

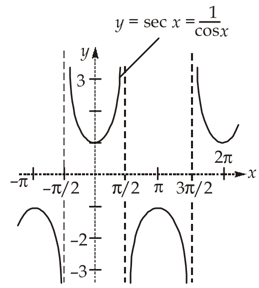
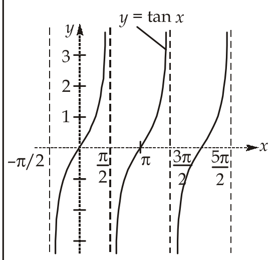
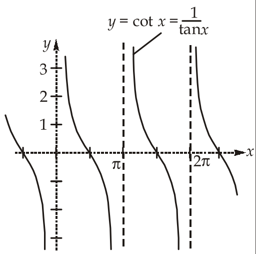
Example:

(a) $\sin(\pi+\theta) = -\sin\theta$ (b) $\tan(\pi-\theta) = -\tan\theta$

(c) $\cos\left(\frac{3\pi}{2}+\theta\right) = \sin\theta$ (d) $\cot\left(\frac{\pi}{2}+\theta\right) - \tan\theta$

4. Properties

Function	Domain	Range	Period	Zeros	Graph
1. $\sin x$	$-\infty < x < \infty$	$-1 \le y \le 1$	2π	$\sin x = 0$ *iff* $x = n\pi$ n is integer	
2. $\cos x$	$-\infty < x < \infty$	$-1 \le y \le 1$	2π	$\cos x = 0$ *iff* $x = (2n+1)\frac{\pi}{2}$ n is integer	
3. $\csc x$	$x \ne n\pi$	$y \le -1$ or $y \ge 1$	2π	No zeros	

Function	Domain	Range	Period	Zeros	Graph
4. sec x	$x \neq (2n+1)\frac{\pi}{2}$	$y \leq -1$ or $y \geq 1$	2π	No zeros	
5. tan x	$x \neq (2n+1)\frac{\pi}{2}$	$-\infty < y < \infty$	π	$\tan x = 0$ *iff* $x = n\pi$ n is integer	
6. cot x	$x \neq n\pi$	$-\infty < y < \infty$	π	$\cot x = 0$ *iff* $x = (2n+1)\frac{\pi}{2}$ n is integer	

5. Important Identities

(i) Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

(ii) Even-Odd Identities

$$\sin(-x) = -\sin x$$

$$\cos(-x) = \cos x$$

$$\tan(-x) = -\tan x$$

(iii) Sum-Difference Formulas

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$

$$\tan(x+y) = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$$

$$\tan(x-y) = \frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$$

(iv) Double Angle Formulas

$$\sin 2x = 2\sin x \cos y$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$= 2\cos^2 x - 1$$

$$= 1 - 2\sin^2 x$$

$$\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}$$

$$\sin(2x) = \frac{2\tan(x)}{1+\tan^2(x)}$$

$$\cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}$$

(v) Power-Reducing/Half Angle Formulas

$$\sin^2(x) = \frac{1-\cos(2x)}{2}$$

$$\cos^2(x) = \frac{1+\cos(2x)}{2}$$

$$\tan^2(x) = \frac{1-\cos(2x)}{1+\cos(2x)}$$

(vi) Product-to-Sum Formulas

$$\sin(x)\sin(y) = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$$

$$\cos(x)\cos(y) = \frac{1}{2}\left[\cos(x-y) + \cos(x+y)\right]$$

$$\sin(x)\cos(y) = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$$

$$\cos(x)\sin(y) = \frac{1}{2}\left[\sin(x+y) - \sin(x-y)\right]$$

(vii) Sum-to-Product Formulas

$$\sin(x)+\sin(y)=2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin(x)-\sin(y)=2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$$

$$\cos(x)+\cos(y)=2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\cos(x)-\cos(y)=-2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

(viii) Multiple Angle Formulas

$$\sin 3x = 3\sin x - 4\sin^3 x$$

$$\cos 3x = 4\cos^3 x - 3\cos x$$

$$\tan 3x = \frac{3\tan x - \tan^3 x}{1-3\tan^2 x}$$

(xi) Relations to Hyperbolic functions

$$\sin ix = i\sin hx$$

$$\cos ix = \cos hx$$

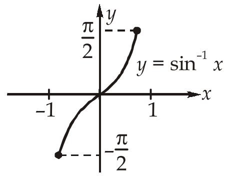
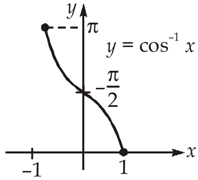
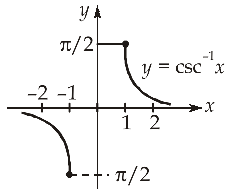
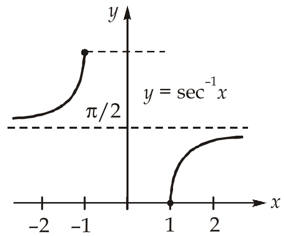
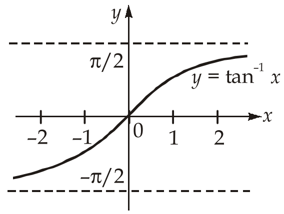
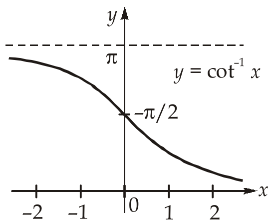
$$\sec ix = \sec hx$$

$$\csc ix - i\csc hx$$

$$\tan ix = i\tan hx$$

$$\cot ix = -i\cot hx$$

6. Inverse Trigonometric Functions

Function	Domain	Range	Graph
1. $y = \sin^{-1} x$ iff $\sin y = x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$	
2. $y = \cos^{-1} x$ iff $\cos y = x$	$-1 \le x \le 1$	$0 \le y \le \pi$	
3. $y = \csc^{-1} x$ iff $\csc y = x$	$x \le -1$ or $x \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2},\ y \ne 0$	
4. $y = \sec^{-1} x$ iff $\sec y = x$	$x \le -1$ or $x \ge 1$	$0 \le y \le \pi,\ y \ne \frac{\pi}{2}$	
5. $y = \tan^{-1} x$ iff $\tan y = x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$	
6. $y = \cot^{-1} x$ iff $\cot y = x$	$-\infty < x < \infty$	$0 < y < \pi$	

VI Theory of Equations

1. Linear Equation in One Variable

$$ax + b = 0$$

$$x = -\frac{b}{a}$$

2. Quadratic Equation

$$ax^2 + bx + c = 0$$

Roots: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Discriminant : $D = b^2 - 4ac$

If $D > 0$ then Roots are real and distinct

If $D = 0$ then Roots are real and equal

If $D < 0$ then Roots are complex conjugates

Algebraic equation: An equation $f(x) = 0$ where $f(x)$ contains algebraic functions (Polynomial, Rational functions).

Transcendental equation: An equation $f(x) = 0$ where $f(x)$ contains non-algebraic functions (Trigonometric, Exponential, Logarithmic, Hyperbolic functions).

3. Polynomial Equations

A polynomial equation has the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0$$

Results

(i) a is root of $f(x) = 0$ if and only if $f(a) = 0$.

(ii) **Factor Theorem:** If a is root of $f(x) = 0$ if and only if $(x - a)$ is factor of $f(x)$.

(iii) Every polynomial equation of nth degree has exactly n roots (real or imaginary).

(iv) Every polynomial equation of odd degree has atleast one real root.

(v) **Descartes' rule of sign:** A polynomial equation $f(x) = 0$ cannot have more positive roots than there are changes of sign in $f(x)$, and cannot have more negative roots than there changes of sign in $f(-x)$. i.e., in a polynomial equation $f(x) = 0$.

(a) *Number* of positive real roots ≤ Number of sign changes in coefficient of $f(x)$.

(b) *Number* of negative real roots ≤ Number of sign changes in coefficients of $f(-x)$.

Note:

☑ If all the coefficients are positive then the equation has no positive real root.

☑ If the coefficients of even powers of x are all of one sign, and the coefficients of the odd powers are all of opposite sign, then the polynomial equation $f(x) = 0$ has no negative real root.

☑ If the equation contains **only even** powers of x and the coefficients are all of the same sign, the equation has no real root.

☑ If the equation contains **only odd** powers of x, and the coefficients are all of the same sign, the equation has no real root except $x = 0$.

4. In a Polynomial Equation

$$a_0x^n + a_1x^{n-1} + .. + a_{n-1}x + a_n = 0$$

Sum of roots = $-\dfrac{a_1}{a_0}$

Sum of the products of the roots taken two at a time = $(-1)^2 \dfrac{a_2}{a_0}$

Sum of the products of the roots taken two at a time = $(-1)^3 \dfrac{a_3}{a_0}$

....

Product of roots = $(-1)^n \dfrac{a_n}{a_0}$

■■■■